NOTES ON SYMPLECTIC GEOMETRY

KELLER VANDEBOGERT

Abstract. These notes are intended to be an example oriented guide in symplectic geometry. They are by no means an introduction, with many fundamental definitions and theorems assumed. The notation is not completely standard, and comes heavily from Souriau's book "Structure of Dynamical System: A Symplectic View of Physics." The material of these notes was motivated and proofread by Francois Ziegler (Georgia Southern University).

Notation: All manifolds and Lie groups are understood to be Hausdorff and countable at ∞ (they may be disconnected). If G is a Lie group, its Lie algebra will be denoted $\mathfrak{g}$. If G acts on a manifold X, then there is an induced infinitesimal action by the Lie algebra $\mathfrak{g}$, defined by

$$Z(x) = \frac{d}{dt}\exp(tZ)x|_{t=0}$$

We use

$$G(x), \quad \mathfrak{g}(x), \quad G_x, \quad \mathfrak{g}_x$$

to denote the G-orbit of x, its tangent space at x, the stabilizer of x in G, and the stabilizer of x in $\mathfrak{g}$. We will have a concise and understood notation for actions of G on T_qG and T_q^*G.

When $\delta q \in T_qG$, we have that (L_g denotes the left action)

$$g\delta q = L_{g*}(q)(\delta q)$$

And similarly, given $p \in T_q^*G$, define gp to be the map such that

$$\langle gp, \delta q\rangle := \langle p, L_{g^{-1}*}(q)(\delta q)\rangle = \langle p, g^{-1}\delta q\rangle$$

Date: September 3, 2017.

Similarly, right actions are defined by the above with L_g replaced by the right action R_g. Finally, the coadjoint action will be denoted $g(p) = gpg^{-1}$, $p \in T_q^*G$.

1. Theorem of Kirillov-Kostant-Souriau

We begin with a theorem that gives a strong characterization of symplectic manifolds. The section after this will use this to classify coadjoint orbits and give these orbits a Hamiltonian G-space structure.

Definition 1.1. The Maurer-Cartan 1-form on $\mathrm{GL}(n)$ is defined by

$$\Theta(\delta g) = g^{-1}\delta g$$

where $\delta g \in T_g\mathrm{GL}(n)$.

Proposition 1.2. *If d denotes the exterior derivative, we have*

$$d\Theta(\delta g, \delta' g) = [g^{-1}\delta g, g^{-1}\delta' g]$$

($[\cdot,\cdot]$ denotes the Lie Bracket)

Proof. This follows immediately. Recall that by deriving $g \cdot g^{-1} = \mathrm{Id}$, we find $\delta(g^{-1}) = -g^{-1} \cdot \delta g \cdot g^{-1}$. Hence,

$$\begin{aligned} \mathrm{d}\Theta(\delta g, \delta' g) &= \delta(g^{-1})\delta' g - \delta'(g^{-1})\delta g \\ &= -g^{-1} \cdot \delta g \cdot g^{-1}\delta' g + g^{-1} \cdot \delta' g \cdot g^{-1}\delta g \\ &= [g^{-1}\delta g, g^{-1}\delta' g] \end{aligned} \tag{1.1}$$

as asserted. ☐

Theorem 1.3. *[Kirillov-Kostant-Souriau]*

(1) *Every coadjoint orbit X of a Lie group G is a homogeneous symplectic manifold when endowed with the KKS 2-form defined*

by

$$\sigma(Z(x), Z'(x)) = \langle x, [Z', Z] \rangle$$

where $x \in \mathfrak{g}^*$, Z, $Z' \in \mathfrak{g}$.

(2) *Conversely, every homogeneous symplectic manifold of a connected Lie group is, up to a possible covering, a coadjoint orbit of some central extension of* G.

Proof. We prove (1) first. This will consist of showing that σ is well-defined, closed, and nondegenerate, and that the action of G preserves this form (that is, $g^*\sigma = \sigma$). Also note that we may assume without loss of generality that G is a matrix group.

The fact that σ is well defined follows immediately, as

$$\langle x, [Z', Z] \rangle = \langle Z(x), Z' \rangle = -\langle Z'(x), Z \rangle$$

by recalling $\mathrm{ad}_{Z'} Z = [Z'Z]$. Now, suppose that $\sigma(Z(x), \cdot) = \langle Z(x), \cdot \rangle = 0$ everywhere. We then immediately see that $Z(x)$ must be identically 0, and similarly for the case of $Z'(x)$.

We now proceed to show that this form is closed. Since X is an orbit, $X = G(x_0)$ for some $X_0 \in \mathfrak{g}^*$. Consider the projection $\pi : G \to X$ which takes $g \mapsto g(x_0)$. If we can show that the pullback by this action is closed, then we will deduce immediately that σ is itself closed. Hence, given tangent vectors $\delta g = Zg$, $\delta' g = Z'g$ for Z, $Z' \in \mathfrak{g}$, we compute

$$\begin{aligned}
\pi^*\sigma(\delta g, \delta' g) &= \sigma(g_*(Zg), \pi_*(Z'g)) \\
&= \sigma(Z(\pi(g)), Z'(\pi(g)) \\
&= \langle g(x_0), [Z', Z]\rangle \\
&= \langle x_0, g^{-1}[Z', Z]g\rangle \qquad (1.2) \\
&= \langle x_0, [g^{-1}Z'g, g^{-1}Zg]\rangle \\
&= \langle x_0, [g\delta' g, g^{-1}\delta g]\rangle \\
&= \langle x_0, \mathrm{d}\Theta(\delta g, \delta' g)\rangle
\end{aligned}$$

where Θ is the Maurer-Cartan 1-form as already introduced. Hence, we see $\pi^*\sigma = \mathrm{d}\langle x_0, \Theta\rangle$. This form is closed and exact, and by naturality of pullbacks and exterior differentiation, $\mathrm{d}\pi^*\sigma = \pi^*\mathrm{d}\sigma = 0$, that is, $\mathrm{d}\sigma = 0$.

It remains to show that σ is G-invariant. First, observe (L_g denotes the left-action of g)

$$\begin{aligned}
g_*(Z(x)) &= DL_g\left(\frac{d}{dt}e^{Zt}x|_{t=0}\right) \\
&= \frac{d}{dt}(ge^{tZ}g^{-1})|_{t=0}(g(x)) \\
&= (\mathrm{Ad}_g Z)(g(x))
\end{aligned}$$

Using this,

$$\begin{aligned}
(g^*\sigma)(Z(x), Z'(x)) &= \sigma(g_*(Z(x)), g_*(Z'(x)) \\
&= \sigma((\mathrm{Ad}_g Z)(g(x)), (\mathrm{Ad}_g Z')(g(x))) \\
&= \langle g(x), [\mathrm{Ad}_g Z', \mathrm{Ad}_g Z]\rangle \qquad (1.3) \\
&= \langle x, \mathrm{Ad}_{g^{-1}}\mathrm{Ad}_g[Z', Z]\rangle \\
&= \langle x, [Z', Z]\rangle = \sigma(Z(x), Z'(x))
\end{aligned}$$

So this action preserves σ.

Conversely, suppose that (X, σ) is a symplectic manifold with a transitive, σ-preserving groups action by G. Then, the induced infinitesimal action preserves σ, that is, $i_Z\sigma = 0$. Since these forms are closed, they are locally exact, and hence up to some coverings of X and G, we may assume these form are exact. This is because there exist universal coverings for X and G for which the above 1-forms will pull back closed forms. However, on a simply connected space, any locally exact form is globally exact. Therefore there exists a moment map, however it is not necessarily equivariant.

Consider the following construction: let $\overline{\mathfrak{g}}$ denote pair $(h, Z) \in C^\infty(X) \times \mathfrak{g}$ such that $Z = \mathrm{drag} h$ ($\mathrm{drag} h$ denotes symplectic gradient). We have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \overline{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

where second map denotes the second projection on $\mathfrak{g}$, and our bracket is $[(h, Z), (h', Z')] = (\{h, h'\}, [Z, Z'])$. The corresponding Lie group $\overline{G}$ will act on X via the action by G and we have moment map Φ defined by $\langle \Phi(x), (h, Z) \rangle = h$. Then, by definition

$$\begin{aligned} i_Z\sigma &= i_{\mathrm{drag} h}\sigma \\ &= -\mathrm{d}h \\ &= -\mathrm{d}\langle \Phi(x), Z \rangle \end{aligned}$$

So that Φ is indeed a moment map. By construction,

$$\begin{aligned} \{\langle \Phi(x), (h, Z) \rangle, \langle \Phi(x), (h', Z') \rangle\} &= \{h, h'\} \\ &= \langle \Phi(x), [(Z, h), Z', h')] \rangle \end{aligned} \tag{1.4}$$

Yielding equivariance, since the above by definition says

$$\sigma(\overline{Z}(x), \overline{Z}'(x)) = \langle \Phi(x), [\overline{Z}, \overline{Z}'] \rangle$$

So that $\sigma = \Phi^*\sigma_{\mathrm{KKS}}$ (see 1.5 below). Finally, using the fact that σ is nondegenerate, we have that $\langle D\Phi(x)(\delta g), \mathfrak{g}\rangle = \sigma(\delta g, \mathfrak{g}(x))$, so that $D\Phi(x)$ has trivial kernel. But then $\Phi : G(x) \to G(\Phi(x))$ is an immersion, and hence a local homeomorphism, giving that Φ is a covering map between these two homogeneous spaces.

□

We have an obvious corollary of the above proof:

Corollary 1.4. *If G is a matrix group with $x_0 \in \mathfrak{g}^*$, then when $x = g(x_0)$ for $x \in X = G(x_0)$:*

$$\begin{aligned} \sigma(\delta x, \delta' x) &= \langle x_0, \delta(g^{-1})\delta' g - \delta'(g^{-1})\delta g\rangle \\ &= \langle x, \delta g \delta'(g^{-1}) - \delta' g \delta(g^{-1})\rangle \end{aligned} \tag{1.5}$$

Proposition 1.5. *Given any equivariant moment map Φ on a Hamiltonian G-space (X, σ, Φ), we have*

$$\sigma = \Phi^*\sigma_{KKS}$$

where σ_{KKS} is the 2-form of 1.3.

Proof. By definition of moment map, we have that for Z, $Z' \in \mathfrak{g}$, $\sigma(Z(x), Z'(x)) = -\mathrm{d}\langle \Phi(x)(Z'(x)), Z(x)\rangle$. Using this with equivariance:

$$\begin{aligned} \sigma(Z(x), Z'(x)) &= -\mathrm{d}\langle \Phi(x)(Z'(x)), Z(x)\rangle \\ &= -\langle Z'(\Phi(x)), Z\rangle \\ &= \langle \Phi(x), [Z', Z]\rangle \\ &= \sigma_{\mathrm{KKS}}(Z(\Phi(x)), Z'(\Phi(x))) \\ &= \sigma_{\mathrm{KKS}}(\Phi_*(Z(x)), \Phi_*(Z(x))) \\ &= \Phi^*\sigma_{\mathrm{KKS}}(Z(x), Z'(x)) \end{aligned} \tag{1.6}$$

whence the result.

□

We now proceed to compute examples using the above theorem.

2. Examples

For given group G, identify all coadjoint orbits and express as a Hamiltonian G-space.

Special Orthogonal Group.

Identifying Orbits: Set $G = SO(3)$. The Lie Algebra is $\mathfrak{g} = \{j(\alpha) \mid \alpha \in \mathbb{R}^3\}$, found by deriving the identity $g\overline{g} = I$.

Note that $\mathfrak{g} \cong \mathbb{R}^3$, so using the standard basis e_i we associate to $\mathfrak{g}^* \cong \mathbb{R}^3$ the dual basis e^i. Then, our dual pairing simply becomes the inner product:

$$\langle \ell, j(\alpha) \rangle = \langle \ell, \alpha \rangle$$

To find the coadjoint action of G, we first need to show that $\mathrm{Ad}_g(j(\alpha)) = j(g\alpha)$. First note that given $g \in SO(3)$ and arbitrary u, v, $w \in \mathbb{R}^3$, we see:

$$\langle gu, g(v \times w) \rangle = \mathrm{vol}(gu, gv, gw) = \langle gu, (gv) \times (gw) \rangle$$

Hence, if $g \in SO(3)$, g distributes over cross products. Using this, given $\beta \in \mathbb{R}^3$:

$$\begin{aligned} \mathrm{Ad}_g(j(\alpha))\beta &= g\big(\alpha \times (\overline{g}\beta)\big) \\ &= (g\alpha) \times \beta = j(g\alpha)\beta \end{aligned}$$

Since $\beta \in \mathbb{R}^3$ was arbitrary, we deduce that $\mathrm{Ad}_g(j(\alpha)) = j(g\alpha)$. Once we have this, the coadjoint action is easily computed. We see:

$$\langle g(\ell), j(\alpha)\rangle = \langle \ell, \mathrm{Ad}_{g^{-1}}(j(\alpha))\rangle = \langle \ell, j(\overline{g}\alpha)\rangle = \langle \ell, \overline{g}\alpha\rangle$$

and the final equality is simply $\langle g\ell, \alpha\rangle$. Hence, the coadjoint action is just matrix multiplication. Our orbits are computed easily of the form $G(se_3)$ for $s \in \mathbb{R}^+$ and the trivial orbit $\{0\}$. Explicitly, these are spheres of radius s.

Specifying Hamiltonian G-space: Now we want to find our 2-form σ. Given $x \in G(se_3)$, set $x = su$ for $u \in S^2$. Deriving $e^{tj(\alpha)}u$ and $t = 0$ gives the infinitesimal action as $j(\alpha)u = \alpha \times u$.

For tangent vectors δu, $\delta' u$ to u, we can set $\alpha = u \times \delta u$ and $\alpha' = u \times \delta' u$. We find:

$$\begin{aligned}
\sigma(j(\alpha)(u), j(\alpha')(u)) &= \langle x, [j(\alpha'), j(\alpha)]\rangle \\
&= \langle su, j((u \times \delta' u) \times (u \times \delta u))\rangle \\
&= s\langle u, j(\mathrm{vol}(u, \delta' u, \delta u)u)\rangle \\
&= s\mathrm{vol}(u, \delta' u, \delta u)
\end{aligned}$$

Hence, our 2-form on each orbit $G(se_3)$ is merely $\sigma(\delta u, \delta' u) = -s\mathrm{vol}(u, \delta u, \delta' u)$. It remains to find a moment map in order to completely specify this as a Hamiltonian G-space, however we can just take this to be the inclusion map.

To see this, if $\Phi(x) = x$, when we only need show $i_Z\sigma = -\mathrm{d}\langle\Phi(x), Z\rangle$. More precisely:

$$\sigma(Z(x), \cdot) = -\langle \cdot, Z\rangle$$

We can of course find $Z' \in \mathfrak{g}$ such that $y = Z'(x)$ so that

$$\begin{aligned}\langle y, Z\rangle &= \langle Z'(x), Z\rangle = \langle \mathrm{ad}_{Z'}x, Z\rangle \\ &= \langle x, -\mathrm{ad}_{Z'}Z\rangle \\ &= -\langle x, [Z', Z]\rangle = \sigma(Z(x), Z'(x)) = \sigma(Z(x), y)\end{aligned}$$

As desired. Thus, our Hamiltonian G-space is the triple $(S^2, \sigma, \mathrm{Id})$.

Special Euclidean Group.

Identifying Orbits: Take $G = \mathrm{SE}(3)$. We know that the Lie Algebra of the Euclidean Group consists of tuples of the form $Z = (j(\alpha), \gamma) \in \mathbb{R}^6$. We identity the dual $\mathfrak{g}^*$ by elements of the form $x = (\ell, p) \in \mathbb{R}^6$. We can define our action by choosing a dual basis similar to the previous problem:

$$\langle x, Z\rangle := \langle l, \alpha\rangle + \langle p, \gamma\rangle$$

Then we can find our coadjoint action by G. Decompose our $g \in \mathrm{SE}(3)$ first as

$$g = \begin{pmatrix} I & c \\ 0 & 1\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix}$$

Then, compute the action for both matrices in the above product. We find:

$$\left\langle \begin{pmatrix} I & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} I & -c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & c \\ 0 & 1 \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} j(\alpha) & \alpha \times c + \gamma \\ 0 & 0 \end{pmatrix} \right\rangle$$
$$= \langle \ell, \alpha \rangle + \langle p, \alpha \times c + \gamma \rangle$$
$$= \langle \ell, \alpha \rangle + \langle c \times p, \alpha \rangle + \langle p, \gamma \rangle \quad \text{(triple product)}$$
$$= \langle \ell + c \times p, \alpha \rangle + \langle p, \gamma \rangle \tag{2.1}$$

Likewise:

$$\left\langle \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} \overline{A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} j(\alpha) & \gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} \overline{A} j(\alpha) A & \overline{A}\gamma \\ 0 & 0 \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} \ell \\ p \end{pmatrix}, \begin{pmatrix} j(\overline{A}\alpha) & \overline{A}\gamma \\ 0 & 0 \end{pmatrix} \right\rangle$$
$$= \langle \ell, \overline{A}\alpha \rangle + \langle p, \overline{A}\gamma \rangle$$
$$= \langle A\ell, \alpha \rangle + \langle Ap, \gamma \rangle \tag{2.2}$$

Hence, composing both of the above coadjoint actions yields $(A, c) \cdot (\ell, p) = (A\ell + c \times Ap, Ap)$. Let $s \in \mathbb{R}$ and $t \in \mathbb{R}^+$ and consider the orbit $G(se_3, ke_3)$. Given a pair $(u, r) \in TS^2$ we can complete u to an orthogonal matrix $A = (w \; v \; u)$ and set $c = r$. We then see that our orbit becomes:

$$s \begin{pmatrix} u \\ 0 \end{pmatrix} + k \begin{pmatrix} r \times u \\ u \end{pmatrix} \in TS^2$$

So this orbit is just a subset of TS^2. If we set $k = 0$, then our orbit is instead a copy of S^2, and if both constants are 0 then we get the 0 orbit.

To see precisely why the above is a copy TS^2, take any element in the orbit $G(x_0)$. It is of the form $(c \times sAe_3 + kAe_3, tAe_3)$. Set $u := Ae_3$ and $r := c - Ae_e\overline{Ae_3}c$. Obviously $||u|| = 1$, and we also see that

$$\langle u, r\rangle = \langle Ae_3, c\rangle - \langle Ae_3, Ae_3\rangle\langle Ae_3, c\rangle = 0$$

So that $(u, r) \in TS^2$. We also have:

$$\begin{pmatrix} r \times su + ku \\ ku \end{pmatrix} = \begin{pmatrix} c \times sAe_3 - Ae_3\langle Ae_3, c \times Ae_3\rangle + kAe_3 \\ kAe_3 \end{pmatrix} = \begin{pmatrix} c \times sAe_3 + kAe_3 \\ kAe_3 \end{pmatrix}$$

and the above is precisely $g(x_0)$, so $G(x_0) \cong TS^2$.

Specifying Hamiltonian G-space: In order to find our 2-form, we will use the following consequence of the KKS Theorem:

$$\sigma(\delta x, \delta' x) = \langle x_0, \delta(g^{-1})\delta' g - \delta'(g^{-1})\delta g\rangle$$

if $x = g(x_0)$. We have:

$$\delta(g^{-1}) = \delta \begin{pmatrix} \overline{A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & -c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \overline{\delta A} & -\overline{A}\delta c - \overline{\delta A}c \\ 0 & 0 \end{pmatrix}$$

Now, $x_0 = (se_3, ke_3)$, and if $g(x_0) = x$, we associate the element g such that $Ae_3 = u$ and $r \in \mathbb{R}^3$ such that $\langle u, r\rangle = 0$ (using the correspondence with TS^2 of the previous part). We compute:

$$\sigma(\delta x, \delta' x) = \left\langle \begin{pmatrix} se_3 \\ ke_3 \end{pmatrix}, \begin{pmatrix} \overline{\delta A} & -\overline{A}\delta r - \overline{\delta A} r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta' A & \delta' r \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \overline{\delta' A} & -\overline{A}\delta' r - \overline{\delta' A} r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta A & \delta r \\ 0 & 0 \end{pmatrix} \right\rangle$$
$$= \left\langle \begin{pmatrix} se_3 \\ ke_3 \end{pmatrix}, \begin{pmatrix} \overline{\delta A}\delta' A - \overline{\delta' A}\delta A & \overline{\delta A}\delta' r - \overline{\delta' A}\delta r \\ 0 & 0 \end{pmatrix} \right\rangle$$
$$= \langle se_3, \delta u \times \delta' u \rangle + \langle ke_3, \overline{\delta A}\delta' r - \overline{\delta' A}\delta r \rangle$$
$$= s\langle u, \delta' u \times \delta u \rangle + \langle k\delta(Ae_3), \delta' r \rangle - \langle k\delta'(Ae_3), \delta r \rangle$$
$$= s\langle u, \delta' u \times \delta u \rangle + k\big(\langle \delta u, \delta' r \rangle - \langle \delta' u, \delta r \rangle\big)$$

Hence, our 2-form is $\sigma(\delta x, \delta' x) = s\langle u, \delta' u \times \delta u \rangle + k\big(\langle \delta u, \delta' r \rangle - \langle \delta' u, \delta r \rangle\big)$. If we set $k = 0$, our orbit is S^2 and we get the 2-form $s\langle u, \delta' u \times \delta u \rangle$, as expected from the previous problem.

Using this gives our Hamiltonian G-space (TS^2, σ, Φ), with the moment map being the map $\Phi : TS^2 \to \mathfrak{g}^*$ sending $(r, u) \mapsto (r \times su + ku, ku)$.

G is any Abelian Group.

Identifying Orbits: When G is an Abelian Lie group, we will have trivial commutator $[g, h] = 0$ for all $g, h \in G$. We can then take $\mathfrak{g} = G$ with our exponential map just being the identity. The infinitesimal action is just $Z(g) = Z + g$, since we have $Z(g) = \frac{d}{dt}\big|_0 tZ + g = Z + g$.

From here, the coadjoint action is simple. We have:

$$\langle g(x), Z \rangle = \langle x, \mathrm{Ad}_{g^{-1}}(Z) \rangle = \langle x, Z \rangle$$

Hence the coadjoint action is trivial: $g(x) = x$ for all $g \in G$, and our orbits are just singleton sets $\{x\}$, $x \in G$.

Specifying Hamiltonian G-space: Since our orbits are merely singleton sets, we have trivial 2-form $\sigma(\delta x, \delta' x) = 0$.

3. Symplectic Induction

Notation: X will always denote a coadjoint orbit of the Lie group G

Definition 3.1. We start with a quick note on convention. We have the natural association $G \times \mathfrak{g}^* = T^*G$. We consider the former set with the left trivialization. That is,

$$\begin{aligned} S :& G \times \mathfrak{g}^* \to T^*G \\ &(q, x) \mapsto qx \end{aligned} \tag{3.1}$$

The above allows us to translate actions on the cotangent bundle to actions on $G \times \mathfrak{g}^*$. Note that the above gives that $S^{-1}(p) = (q, q^{-1}p)$ when $p \in T_q^*G$. The left trivialization gives a left action:

$$\begin{aligned} g(q, x) &= S^{-1}(g(S(q, x))) \\ &= S^{-1}(gqx) \\ &= (gq, (gq)^{-1}(gqx)) \\ &= (gq, x) \end{aligned}$$

Where we noted that $gqx \in T_{gq}^*G$ in order to take the inverse image. Similarly, consider the right action by the inverse, that is, $g(p) := pg^{-1}$:

$$\begin{aligned} g(x, q) &= S^{-1}(g(S(qx))) \\ &= S^{-1}(qxg^{-1}) \\ &= (qg^{-1}, qg^{-1})^{-1}(qxg^{-1})) \\ &= (qg^{-1}, g(x)) \end{aligned} \tag{3.2}$$

In order to build intuition, consider a Lie group G and its coadjoint action on $\mathfrak{g}^*$. Given a subgroup $A \leqslant G$, we want to consider conditions on A making the following diagram commute

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*"] \arrow[d, "\pi"'] & \mathfrak{g}^* \arrow[d, "\pi"] \\ \mathfrak{a}^* \arrow[r, "g(\cdot)"] & \mathfrak{a}^* \end{tikzcd} \tag{3.3}$$

In order for an action $G\times\mathfrak{a}^* \to \mathfrak{a}^*$ to exist, we must have that $\mathrm{Ad}_G(\mathfrak{a}) = \mathfrak{a}$, that is, A must be normal.

Definition 3.2 (Symplectic Induction)**.** Consider a closed subgroup $H \leqslant G$ and a Hamiltonian H-space (Y, τ, Ψ). We can produce a Hamiltonian G-space as follows.

First, set $N := T^*G \times Y$ endowed with 2-form $\omega = \mathrm{d}\sigma + \tau$ where σ denotes the canonical 1-form. Let H act of N by setting

$$h(p, y) := (ph^{-1}, h(y))$$

We then have moment map

$$\psi(p, y) = \Psi(y) - q^{-1}p|_{\mathfrak{h}}$$

We can define an induced manifold structure on

$$\mathrm{Ind}_H^G Y := \psi^1(0)/H$$

We have that ψ is a submersion, so that $\psi^{-1}(0)$ is a submanifold, and the quotient by H is a manifold. The 2-form $\omega|_{\psi^{-1}(0)}$ vanishes precisely on the orbits of H, so that ω descends onto some nondegenerate form ω_{ind}.

To give this a G-space structure, we can define the action of G on N by $g(p, y) = (gp, y)$. This commutes with the above H action, and the moment map can be given as

$$\phi(p, y) = pq^{-1}$$

where $p \in T_q^*G$. This is constant on H orbits, and hence passing to the quotient gives the G-action and moment map Φ_{ind}. To get a handle on the spaces involved, we have the following commutative diagram

$$\begin{tikzcd} \psi^{-1}(0) \arrow[r] \arrow[d] & N \arrow[d, "\phi"] \\ \mathrm{Ind}_H^G Y \arrow[r, "\Phi_{\text{ind}}"] & \mathfrak{g}^* \end{tikzcd} \tag{3.4}$$

The diagram (3.4) gives us

Proposition 3.3. *Let W be any orbit in $\mathfrak{g}^*$. Then,*

$$W \cap \mathrm{Im}(\Phi_{ind}) \neq \varnothing \iff W|_{\mathfrak{h}} \cap \mathrm{Im}(\Psi) \neq \varnothing$$

Proof. Suppose first that $W \cap \mathrm{Im}(\Phi_{\text{ind}} \neq \varnothing$. Choose $w \in W \cap \mathrm{Im}(\Phi_{\text{ind}}$, so that $w = pq^{-1}$ for some pair $(p, y) \in \psi^{-1}(0)$, $p \in T_q^*G$.

Then, by definition, $\Psi(y) = q^{-1}p|_{\mathfrak{h}}$. The above shows that $p = wq$, so that $\Psi(y) = q^{-1}(w)|_{\mathfrak{h}} \in W|_{\mathfrak{h}}$. Hence, $W|_{\mathfrak{h}} \cap \mathrm{Im}(\Psi) \neq \varnothing$.

Conversely, choose $w|_{\mathfrak{h}} \in W|_{\mathfrak{h}} \cap \mathrm{Im}(\Psi)$. This means that $w|_{\mathfrak{h}} = \Psi(y)$ for some $y \in Y$, so that the pair $(qw, y) \in \psi^{-1}(0)$. Taking the image modulo H and then by Φ_{ind} gives us

$$\Phi_{\text{ind}}(qw, y) = q(w) \in W$$

Completing the proof.

□

Definition 3.4. $\mathrm{Ann}(X)$ denotes the set of all $Z \in \mathfrak{g}$ such that $\langle X, Z\rangle = 0$, with $\langle \cdot, \cdot \rangle$ denoting the dual pairing.

Definition 3.5. A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ will be called X-abelian if $[\mathfrak{a}, \mathfrak{a}] \subset \mathrm{Ann}(X)$.

Definition 3.6. A subgroup A of the Lie group G will be called X-abelian if A is a closed, connected subgroup with X-abelian Lie algebra $\mathfrak{a}$.

Proposition 3.7. *Let G be a Lie Group with A a normal X-abelian subgroup. Then, we have an action of G on $\mathfrak{a}^*$ making $\pi : \mathfrak{g}^* \to \mathfrak{a}^*$ equivariant, implying that $X|_{\mathfrak{a}}$ is also a coadjoint orbit of G. That is,*

$$X|_{\mathfrak{a}} = G(p) = G/H$$

where $X = G(x)$, $p = x|_{\mathfrak{a}}$, and H is the stabilizer of p. In this case, there is a unique coadjoint orbit Y of H (namely, $Y = H(x|_{\mathfrak{h}})$) such that

$$(a)\ X = Ind_H^G Y \quad (b)\ Y|_{\mathfrak{a}} = \{p\}$$

Moreover, Y is the reduced space $\pi^{-1}(p)/A$, $\pi : X \to \mathfrak{h}^*$ is the natural projection.

Conversely, any H-orbit Y in $\mathfrak{h}^*$ satisfying (b) is such that $Ind_H^G Y$ is isomorphic to some coadjoint orbit of $\mathfrak{g}^*$.

Before proving this, we will need

Lemma 3.8. *In the above assumptions, we have*

(1) $\mathfrak{a}(x) = \mathrm{Ann}(\mathfrak{h})$
(2) $A(x) = x + \mathrm{Ann}(\mathfrak{h})$
(3) $H(x) = \eta^{-1}(H(x|_{\mathfrak{h}}))$, *$\eta : \mathfrak{g}^* \to \mathfrak{h}^*$ is the natural projection.*

Proof. To prove (1), it suffices to show that $\mathrm{Ann}(\mathfrak{a}(x)) = \mathfrak{h}$. Let $Z \in \mathfrak{g}$. We have:

$$\begin{aligned}
\langle \mathfrak{a}(x), Z\rangle &= \langle x, [\mathfrak{a}, Z]\rangle \\
&= \langle x|_{\mathfrak{a}}, [\mathfrak{a}, Z]\rangle \quad \text{since } [\mathfrak{a}, Z] \subset \mathfrak{a} \\
&= \langle Z(p), \mathfrak{a}\rangle
\end{aligned}$$

Hence, $\mathrm{Ann}(\mathfrak{a}(x))$ is precisely the set of $Z \in \mathfrak{g}$ such that $Z(p) = 0$, which gives that $\exp(Z) \in H \implies Z \in \mathfrak{h}$. We then deduce $\mathfrak{a}(x) = \mathrm{Ann}(\mathfrak{h})$.

For part (2), note that the above shows that $\langle \mathfrak{a}(x), \mathfrak{h}\rangle = 0$, so that $\mathfrak{a}$ stabilizes $x|_{\mathfrak{h}}$. Exponentiating yields that A must also stabilize $x|_{\mathfrak{h}}$. Hence, $A(x) \subset x + \mathrm{Ann}(\mathfrak{h})$, as $\langle x + \mathrm{Ann}(\mathfrak{h}), \mathfrak{h}\rangle = 0$.

For the reverse inclusion, given $\exp(Z)(x) \in A(x)$ $(Z \in \mathfrak{a})$, we see for $Z' \in \mathfrak{g}$:

$$\begin{aligned}
\langle \exp(Z)(x), Z'\rangle &= \langle \sum_{n=0}^{\infty} \frac{\mathrm{ad}(Z)^n(x)}{n!}, Z'\rangle \\
&= \langle x, \sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{ad}(Z)^n(Z')}{n!}\rangle \\
&= \langle x, Z' + [Z, Z'] + \sum_{n=2}^{\infty} \frac{(-1)^n \mathrm{ad}(Z)^{n-2}}{n!}[Z, [Z, Z']]\rangle
\end{aligned}$$

Since $\mathfrak{a}$ is an ideal, we see that $[Z, Z'] \in \mathfrak{a}$, and since A is X-abelian, $[Z, [Z, Z']] \in \mathrm{Ann}(X)$, so that

$$\langle x, \sum_{n=2}^{\infty} \frac{(-1)^n \mathrm{ad}(Z)^{n-2}}{n!}[Z, [Z, Z']]\rangle = 0$$

Hence, we see

$$\langle \exp(Z)(x), Z'\rangle = \langle x, Z' + [Z, Z']\rangle = \langle x + Z(x), Z'\rangle$$

So that given $x + Z(x) \in x + \mathrm{Ann}(\mathfrak{h})$, we see that in fact $x + Z(x) = \exp(Z)(x) \in A(x)$, proving equality.

For part (3), note that η explicitly takes $x \mapsto x|_{\mathfrak{h}}$, so that $\mathfrak{g}^*/\operatorname{Ann}(\mathfrak{h}) = \mathfrak{h}^*$. This implies $\eta^{-1}(x) = x + \operatorname{Ann}(\mathfrak{h})$ so that the result of part (2) merely says $A(x) = \eta^{-1}(x|_{\mathfrak{h}})$. Since $A \subset H$, $H \cdot A = H$, and by equivariance, $H(\eta^{-1}(x|_{\mathfrak{h}})) = \eta^{-1}(H(x_{\mathfrak{h}}))$, so that

$$H(x) = \eta^{-1}(H(x_{\mathfrak{h}}))$$

Completing the proof. □

Now we can prove the proposition.

Proof. Note that equivariance has already been proved by the introductory discussion. Hence, we need show that in the setting of our proposition, the orbit is unique.

We need to prove two directions. In the first direction, we need to show that $Y := H(x|_{\mathfrak{h}})$ is such that $\Phi_{\text{ind}} : \operatorname{Ind}_H^G Y \to X$ is a symplectic diffeomorphism with $Y|_{\mathfrak{a}} = \{p\}$. Conversely, we must prove that given any Y satisfying the above two properties, we may deduce that $Y = H(x|_{\mathfrak{h}})$.

We begin by first assuming that $Y := H(x_{\mathfrak{h}})$. It is clear that $Y|_{\mathfrak{a}} = \{p\}$, since we have that $\langle H(x|_{\mathfrak{h}}), \mathfrak{a}\rangle = \langle x|_{\mathfrak{h}}, \operatorname{Ad}_H(\mathfrak{a})\rangle$. Since A is normal, the above merely becomes $x|_{\mathfrak{a}} = p$ (since $\mathfrak{a} \subset \mathfrak{h}$). It remains to show that the definition given for Φ_{ind} above is a symplectic diffeomorphism.

Let us show surjectivity first. To do this, we claim that if any orbit W is such that $W|_{\mathfrak{h}} \cap Y \neq \varnothing$, then $W = X$. This, when combined with 3.3 will imply that $\operatorname{Im}(\Phi_{\text{ind}} = X$.

Proof of claim: Choose $g(w)|_{\mathfrak{h}} \in W|_{\mathfrak{h}} \cap Y$, where $W = G(w)$ for some $w \in \mathfrak{g}^*$. Then, we see that there exists $h \in H$ such that

$$\eta(g(w)) = h(x|_{\mathfrak{h}})$$

So that, employing (3) of 3.8:

$$g(w) \in \eta^{-1}(h(x|_{\mathfrak{h}}) \subset \eta^{-1}(H(x|_{\mathfrak{h}})) = H(x)$$

Therefore, $w = g^{-1}h(x) \in G(x)$, which means that $G(w) = G(x) \implies W = X$.

Using this claim, note that the moment map $\Psi : Y \to X|_{\mathfrak{h}}$ is merely the inclusion map. Hence, $W|_{\mathfrak{h}} \cap Y \neq \varnothing$ if and only if $W \cap \mathrm{Im}(\Phi_{\mathrm{ind}}) \neq \varnothing$. Since this only holds when $W = X$, the only orbit intersecting $\mathrm{Im}(\Phi_{\mathrm{ind}}$ is X, so that in fact $\mathrm{Im}(\Phi_{\mathrm{ind}} = X$, showing surjectivity.

Now, injectivity will follow if we can show that the preimage by Φ_{ind} of every point in X is still a singleton in $\mathrm{Ind}_H^G Y$. Owing to the diagram (3.4), we see that for $x \in X$

$$\Phi_{\mathrm{ind}}^{-1}(x) = (\psi^{-1}(0) \cap \phi^{-1}(x))/H$$

Hence it suffices to show that $\psi^{-1}(0) \cap \phi^{-1}(x)$ consists of a single H-orbit. Now $\phi^{-1}(x)$ is easily computed as $xG \times Y$. Similarly, $\psi^{-1}(0) = \{(p, y) \mid y = q^{-1}p|_{\mathfrak{h}}\}$. Taking the intersection, we see that we must have $p = xq$ and $q^{-1}(x) \in \eta^{-1}(Y)$, so that

$$\psi^{-1}(0) \cap \phi^{-1}(x) = \{(xq, q^{-1}(x)|_{\mathfrak{h}} \mid q \in Q\}$$

Where Q is the set of all $q \in G$ with $q^{-1}(x) \in \eta^{-1}(Y)$. However, this immediately gives that $Q = H$ by (3) of Lemma 3.8, so that the above set is indeed a single orbit. Hence Φ_{ind} is a bijection. By equivariance, the G-action on $\mathrm{Ind}_H^G Y$ is transitive. Using part (2) of 1.3, we deduce that Φ_{ind} is an injective covering, hence trivially a diffeomorphism.

Conversely, assume now that Y is any orbit satisfying the above assumptions. Since X lies in the image of Φ_{ind}, we have that $X|_{\mathfrak{h}} \supset Y$ by Proposition 3.3. This means $y = g(x)|_{\mathfrak{h}}$ for some $x \in X$. Projecting onto $\mathfrak{a}^*$, we see that by assumption (b), $y|_{\mathfrak{a}} = p$, and by equivariance $g(x)|_{\mathfrak{a}} = g(p)$. Therefore $p = g(p)$ so that $g \in H$, giving $Y = H(x)|_{\mathfrak{h}} = H(x|_{\mathfrak{h}})$.

Finally, to prove the last assertion, consider all fibers of the projection $\eta : X \to \mathfrak{h}^*$. Points in the range are of the form $g(x)|_{\mathfrak{h}}$, which says that their preimages are of the form $g(x) + \mathrm{Ann}(\mathfrak{h})$. By part (2) of our Proposition, $\mathrm{Ann}(\mathfrak{h}) = \mathfrak{a}(x)$, and by transitivty of our G-action, $\mathfrak{a}(x) = \mathfrak{a}(g(x))$. However, using the fact that A is normal,

$$\begin{aligned} g(x) + \mathrm{Ann}(\mathfrak{h}) &= g(x) + \mathfrak{a}(g(x)) \\ &= g(x) + g(\mathfrak{a}(x)) \\ &= g(x + \mathfrak{a}(x)) \\ &= g(A(x)) \\ &= A(g(x)) \end{aligned} \tag{3.5}$$

so that every fiber of π is an A-orbit of X. By equivariance of π, we also have that $H(x) = \pi^{-1}(p)$. Using the above, however, yields that $H(x)/A = H(x)|_{\mathfrak{h}}$, that is,

$$\pi^{-1}(p)/A = H(x|_{\mathfrak{h}})$$

Now, in order to prove the converse, it suffices to show that the moment map Φ_{ind} is bijective onto some orbit $G(x)$.

Let us first show surjectivity. Using 3.3, we deduce that

$$\mathrm{Im}\,\Phi_{\mathrm{ind}} = \bigcup \{W \mid W \ G-\mathrm{orbit},\ W|_{\mathfrak{h}} \cap Y \neq \varnothing\}$$

Now suppose we have two orbits $G(w_1)$, $G(w_2)$ in the above set. Then, their restrictions to $\mathfrak{h}$ are nonempty, and we see that $H(w_1|_{\mathfrak{h}}) = H(w_2|_{\mathfrak{h}})$. Using 3.8, this implies that $H(w_1) = H(w_2)$, so that $G(w_1) = G(w_2)$. Therefore $\mathrm{Im}(\Phi_{\mathrm{ind}}) \supset G(w_1)$, and the reverse inequality merely follows by equivariance of the induced moment map. Hence we see $\mathrm{Im}(\Phi_{\mathrm{ind}}) = G(w_1)$, showing surjectivity. Injectivity is proved exactly as above.

To show $p \in G(w_1)|_{\mathfrak{h}}$, merely note that $G(w_1)|_{\mathfrak{a}} = Y|_{\mathfrak{a}} = \{p\}$.

$\square$

4. Example

Computing $\mathbf{Ind}_{\mathbb{C}}^{E(2)}\{P_0\}$. Consider the Euclidean group consisting of matrices

$$\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}$$

with $A \in \mathbb{R}$, $C \in \mathbb{C}$. We have the trivial Lie Algebra identification

$$\begin{pmatrix} i\alpha & \gamma \\ 0 & 0 \end{pmatrix}$$

with $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{C}$. We then associate our dual space $\mathfrak{g}^* = \mathbb{R} \times \mathbb{C}$, so that $T^*G = E(2) \times \mathbb{R} \times \mathbb{C}$, and our group actions will we determined by the left trivialization as outlined previously.

Now, let us compute $\mathrm{Ind}_{\mathbb{C}}^{E(2)}\{P_0\}$ for $P_0 \in \mathbb{C}$. Note that the singleton space is trivially a Hamiltonian G-space with $\Phi(P_0) = P_0$. In this case be have $H = \mathfrak{h} = \mathbb{C}$, which we are identifying with the matrix subgroup

$$\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}$$

We will define our dual pairing as

$$\left\langle \begin{pmatrix} L \\ P \end{pmatrix}, \begin{pmatrix} i\alpha & \gamma \\ 0 & 0 \end{pmatrix} \right\rangle = L\alpha + \mathrm{Re}(\overline{P}C)$$

Computing the 2-form: We must first find out canonical 1-form θ. This is defined by taking the dual to $\langle p, \delta q\rangle$, for $p \in T_q^*G$. Given $(q, x) \in G \times \mathfrak{g}^*$, we identify this with $p = qx$ so that

$$\theta = \langle x, q^{-1}\delta q\rangle$$

We see

$$\delta q = \delta \begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ie^{iA}\delta A & \delta C \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \langle x, q^{-1}\delta q\rangle &= \left\langle \begin{pmatrix} L \\ P \end{pmatrix}, \begin{pmatrix} e^{-iA} & -e^{-iA}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ie^{iA}\delta A & \delta C \\ 0 & 0 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} L \\ P \end{pmatrix}, \begin{pmatrix} i\delta A & e^{-iA}\delta C \\ 0 & 0 \end{pmatrix} \right\rangle \\ &= L\delta\alpha + \mathrm{Re}(\overline{P}e^{-iA}\delta C) \end{aligned} \tag{4.1}$$

Giving our 1-form as $\theta = L\mathrm{d}A + \mathrm{Re}(\overline{Pe^{iA}}\mathrm{d}C)$, and upon taking the exterior derivative, we find our 2-form

$$\sigma = \mathrm{d}L \wedge \mathrm{d}A + \mathrm{Re}(\overline{\mathrm{d}Pe^{iA}} \wedge \mathrm{d}C)$$

Specifying H-action. By the construction outlined for symplectic induction, we have that $\mathbb{C}$ acts on $T^*E(2)$ by $h(p) = ph^{-1}$. Again, by our convention with left trivialization, $(q,x)h^{-1} = qh^{-1}, h(x))$. Hence, we must first find the coadjoint action by $\mathbb{C}$. This is found as

$$\begin{aligned} &\left\langle \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L \\ P \end{pmatrix}, \begin{pmatrix} i\alpha & \gamma \\ 0 & 0 \end{pmatrix} \right\rangle \\ &\qquad = \left\langle \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} L \\ P \end{pmatrix}, \begin{pmatrix} i\alpha & i\alpha C + \gamma \\ 0 & 0 \end{pmatrix} \right\rangle \\ &\qquad = L\alpha - \alpha \mathrm{Im}(\overline{P}C) + \mathrm{Re}(\overline{P}\gamma) \\ &\qquad = \langle (L - \mathrm{Im}(\overline{P}C), P), (i\alpha, \gamma)\rangle \end{aligned}$$

So we have coadjoint action $C(L,P) = (L - \mathrm{Im}(\overline{P}C), P)$. Then, we can make $z \in \mathbb{C}$ act on an element $m \in N = T^*G$ by

$$z(m) = \left(\begin{pmatrix} e^{iA} & z - e^{iA} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L - \mathrm{Im}(\overline{P}z) \\ P \end{pmatrix} \right)$$

And deriving gives the infinitesimal action by $\mathfrak{h} = \mathbb{C}$:

$$Z(m) = \left(\begin{pmatrix} 0 & -e^{iA}Z \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -\mathrm{Im}(\overline{P}Z) \\ 0 \end{pmatrix} \right)$$

Computing the Moment Map: Let us compute the moment map. We already have that $\Psi(P_0) = P_0$, so it remains to find $q^{-1}p|_{\mathfrak{h}}$, $p \in T_q^*G$. This is not difficult, however, as identifying $p = qx$ merely becomes the restriction $x|_{\mathfrak{h}}$. But this is simply (L, P) restricted to P, so that

$$\psi(m, P_0) = P_0 - P$$

But then we can immediately find $\psi^{-1}(0)$ as the set

$$\psi^{-1}(0) = \left\{ \left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right) \mid A, L \in \mathbb{R},\ L \in \mathbb{C} \right\}$$

Restricting our 2-form $\sigma|_{\psi^{-1}(0)}$, we find:

$$\begin{aligned} \sigma|_{\psi^{-1}(0)} &= \mathrm{d}L \wedge \mathrm{d}A + \mathrm{Re}(\overline{\mathrm{d}P_0 e^{iA}} \wedge \mathrm{d}C) \\ &= \mathrm{d}L \wedge \mathrm{d}A + \mathrm{Re}(\overline{iP_0 e^{iA}\mathrm{d}A} \wedge \mathrm{d}C) \\ &= \mathrm{d}L \wedge \mathrm{d}A - \mathrm{Im}(\overline{P_0 e^{iA}}\mathrm{d}C) \wedge \mathrm{d}A \\ &= \mathrm{d}\ell \wedge \mathrm{d}A \end{aligned} \tag{4.2}$$

Where $\ell = L - \mathrm{Im}(\overline{P_0 e^{iA}}C)$.

Giving a G-space Structure to the Reduction: Since our 2-form vanishes precisely along the orbits of H, we can realize the quotient $\psi^{-1}(0)/H$ as $\mathbb{R} \times S^1$ via the identification

$$\left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right) \mapsto \begin{pmatrix} L - \mathrm{Im}(\overline{P_0 e^{iA}}C) \\ e^{iA} \end{pmatrix}$$

We give this a G-space structure via the 2-form already computed as

$$\omega = \mathrm{d}\ell \wedge \mathrm{d}A$$

where ℓ is defined as above. We have the map $\phi : T^*G \to \mathfrak{g}^*$ defined by $\phi(p, y) = pq^{-1}$. Recalling our convention, this gives that $\phi(q, x) = q(x)$.

The coadjoint action is easily computed as above, so that

$$\phi\left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P \end{pmatrix} \right) = \begin{pmatrix} L - \operatorname{Im}(\overline{e^{iA}P}C) \\ e^{iA}P \end{pmatrix}$$

Now, given an element

$$\begin{pmatrix} L - \operatorname{Im}(\overline{P_0 e^{iA}}C) \\ e^{iA} \end{pmatrix} \in \operatorname{Ind}_{\mathbb{C}}^{E(2)}\{P_0\}$$

this lifts to the pair

$$\left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right) \in \psi^{-1}(0)$$

Recalling commutativity of (3.4), we then see that

$$\Phi_{\text{ind}}\left(\begin{pmatrix} L - \operatorname{Im}(\overline{P_0 e^{iA}}C) \\ e^{iA} \end{pmatrix} \right) = \phi\left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right) = \begin{pmatrix} L - \operatorname{Im}(\overline{e^{iA}P_0}C) \\ e^{iA}P_0 \end{pmatrix}$$

Obviously our previously computed 2-form descends to $\omega = \mathrm{d}\ell \wedge \mathrm{d}A$.

The action by G on $\psi^{-1}(0)$ is merely $gp = (gq, x)$, that is,

$$\begin{pmatrix} e^{ia} & c \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} e^{iA} & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right) = \left(\begin{pmatrix} e^{i(A+a)} & e^{ia}C + c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ P_0 \end{pmatrix} \right)$$

Which descends onto the G action (upon taking the quotient)

$$\begin{pmatrix} e^{ia} & c \\ 0 & 1 \end{pmatrix}_{\mathbb{R}\times S^1} \begin{pmatrix} L - \operatorname{Im}(\overline{e^{iA}P_0}C) \\ e^{iA} \end{pmatrix} = \begin{pmatrix} L - \operatorname{Im}(\overline{e^{i(A+a)}P_0}C) \\ e^{i(A+a)} \end{pmatrix}$$

Which completes the construction of $\operatorname{Ind}_{\mathbb{C}}^{E(2)}\{P_0\}$.

5. Another Example

SE(3). It will be illuminating to see how Proposition 3.7 can recover the orbits of SE(3). We start by noting the only normal subgroup is of the form

$$\begin{pmatrix} I & c \\ 0 & 1 \end{pmatrix}$$

For $c \in \mathbb{R}^3$. This is Abelian, with Lie algebra

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

with $\gamma \in \mathbb{R}^3$, and we consider orbits of pairs $x = (se_3, ke_3) \in \mathfrak{a}^*$ with e_3 the third standard basis element. The G-action on $x|_{\mathfrak{a}} = ke_3$ is such that $g(x|_{\mathfrak{a}^*}) = kAe_3$, where

$$g = \begin{pmatrix} A & c \\ 0 & 1 \end{pmatrix}$$

So that the stabilizer is easily computed as all G elements of the form

$$\begin{pmatrix} e^{j(te_3)} & c \\ 0 & 1 \end{pmatrix}$$

that is, rotations about the z axis ($t \in \mathbb{R}$) combined with translations by $c \in \mathbb{R}^3$. Rotations in the plane correspond to elements of SO(2), so that $G_p := H = \mathrm{SO}(2) \times \mathbb{R}^3$. Then, we want to consider the H orbit $H(x|_{\mathfrak{h}})$. This will give, for $h \in H$,

$$\begin{pmatrix} e^{j(te_3)} & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} se_3 \\ ke_3 \end{pmatrix} = \begin{pmatrix} se_3 + kc \times e_3 \\ ke_3 \end{pmatrix}$$

And, upon restricting to our Lie algebra, we see that $\mathfrak{h} = \mathbb{R} \times \mathbb{R}^3$ in the obvious fashion, so that upon restricting, the term $kc \times e_3$ is annihilated and we are left with a singleton. Hence, as $h \in H$ was arbitrary

$$H(x|_{\mathfrak{h}}) = \begin{pmatrix} se_3 \\ ke_3 \end{pmatrix}$$

Computing H-action on the Cotangent Bundle. The H-action is abstractly $h(p, y) = (ph^{-1}, h(y))$. Since our $Y = H(x|_{\mathfrak{h}})$ is merely a singleton, this can be discarded. Recalling our convention, our H-action is $h(q, x) = (qh^{-1}, h(x))$, where $(q, x) \in G \times \mathfrak{g}^*$. The coadjoint action has already been computed previously, so we see

$$\begin{aligned} \begin{pmatrix} e^{j(te_3)} & c \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ p \end{pmatrix} \right) &= \left(\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-j(te_3)} & -e^{j(te_3)}c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{j(te_3)}L + c \times e^{j(te_3)}P \\ e^{j(te_3)}P \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} Ae^{-j(te_3)} & C - Ae^{-j(te_3)}c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{j(te_3)}L + c \times e^{j(te_3)}P \\ e^{j(te_3)}P \end{pmatrix} \right) \end{aligned}$$

And the moment map is computed as the difference

$$\psi(p,y)=\begin{pmatrix} se_3 \\ ke_3 \end{pmatrix}-\begin{pmatrix} \langle L,e_3\rangle e_3 \\ P \end{pmatrix}$$

so that

$$\psi^{-1}(0)=\left\{\left(\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix},\begin{pmatrix} L \\ ke_3 \end{pmatrix}\right) \mid \langle L,e_3\rangle = s\right\}$$

We also have the moment map from T^*G onto $\mathfrak{g}^*$ sending $p \mapsto pq^{-1}$ for $p \in T_q^*G$, that is, $(q,x) \mapsto (1,q(x)) \mapsto q(x)$. Hence, ϕ is just the coadjoint action on $x \in \mathfrak{g}^*$.

Computing canonical Cotangent Bundle 2-form. To compute the 2-form on T^*G, we proceed as done previously. When $p \in T_q^*G$ corresponds to (q,x), we get

$$\langle x, q^{-1}\delta q\rangle$$

We see

$$\delta q=\begin{pmatrix} \delta A & \delta C \\ 0 & 0 \end{pmatrix}$$

So that

$$\langle x, q^{-1}\delta q\rangle = \left\langle \begin{pmatrix} L \\ P \end{pmatrix},\begin{pmatrix} \overline{A}\delta A & \overline{A}\delta C \\ 0 & 0 \end{pmatrix}\right\rangle$$

Which becomes

$$\langle L, j^{-1}(\overline{A}\delta A)\rangle + \langle AP, \delta C\rangle$$

Our 1-form becomes

$$\theta = Lj^{-1}(\overline{A}\mathrm{d}A) + AP\mathrm{d}C$$

So that the 2-form is merely

$$\mathrm{d}\theta = \mathrm{d}L \wedge j^{-1}(\overline{A}\mathrm{d}A) + Lj^{-1}(\overline{\mathrm{d}A}\wedge \mathrm{d}A) + (\mathrm{d}A)P\wedge \mathrm{d}C + A\mathrm{d}P\wedge \mathrm{d}C$$

Which, in tensor form, becomes

$$\mathrm{d}\theta = A_j^i \mathrm{d}L_i \wedge \mathrm{d}A_j^i + L_i \mathrm{d}A_j^i \wedge \mathrm{d}A_j^i + P_i \mathrm{d}A_j^i \wedge \mathrm{d}C_j + A_i^j \mathrm{d}P_i \wedge \mathrm{d}C_j$$

(summation convention). Restricting the above to $\psi^{-1}(0)$, obviously $\mathrm{d}P = 0$, and $\mathrm{d}L_3 = 0$.

Structure of $\mathbf{Ind}_{\mathbf{SO}(2)\times\mathbb{R}^3}^{\mathbf{SE}(3)}(se_3, ke_3)$. We can compute the image of $\psi^{-1}(0)$ mod H by using the previous results on SE(3). We already have a moment map $\Phi : TS^2 \to \mathfrak{g}^*$, so we can consider taking an element mod H, mapping it under Φ_{ind}, and then taking the preimage by our previously computed Φ. These spaces must be symplectically diffeomorphic by 3.7. Using the diagram (3.4), we know that

$$\Phi_{\text{ind}}((p,y) \mod H) = \begin{pmatrix} AL + C \times kAe_3 \\ kAe_3 \end{pmatrix}$$

Now, taking the preimage by Φ gives

$$\Phi^{-1}\left(\begin{pmatrix} AL + C \times kAe_3 \\ kAe_3 \end{pmatrix} \right) = \begin{pmatrix} C - AL\langle AL, C\rangle \\ Ae_3 \end{pmatrix}$$

So we can characterize the reduction modulo H as

$$\left(\begin{pmatrix} A & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} L \\ ke_3 \end{pmatrix} \right) \mapsto \begin{pmatrix} C - AL\langle AL, C\rangle \\ Ae_3 \end{pmatrix}$$

With moment map

$$\Phi_{\text{ind}}\left(\begin{pmatrix} C - AL\langle AL, C\rangle \\ Ae_3 \end{pmatrix} \right) = \begin{pmatrix} AL + C \times kAe_3 \\ kAe_3 \end{pmatrix}$$

And we can easily compute our G-action as

$$\begin{pmatrix} A' & C' \\ 0 & 1 \end{pmatrix}_{TS^2} \begin{pmatrix} C - AL\langle AL, C\rangle \\ Ae_3 \end{pmatrix} = \begin{pmatrix} A'(C - AL\langle AL, C\rangle) + C' - A'AL\langle A'AL, C'\rangle \\ A'Ae_3 \end{pmatrix}$$

Which gives us our desired G-space structure.

6. The Poincaré Group

Definition, Lie algebra, and dual pairing. The Poincaré group is the group of matrices

$$g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

With $L \in \mathrm{SO}(3,1)^o$ (the identity component of the Lorentz group) and $C \in \mathbb{R}^{3,1}$. The Lorentz group is the group of matrices that preserve the Minkowski metric, that is,

$$g(LX, LY) = g(X, Y)$$

where $g(X,Y) = t_X t_Y - xy$ if $X = (t_X, x)$, $Y = (t_Y, y)$. We will define our transpose in a way that $g(LX, Y) = g(X, \overline{L}Y)$, which is explicitly

$$\overline{\begin{pmatrix} A & b \\ c & d \end{pmatrix}} = \begin{pmatrix} A^t & -c^t \\ -b^t & d \end{pmatrix}$$

($A \in M_3(\mathbb{R})$, $b,\ c \in \mathbb{R}^3$, $d \in \mathbb{R}$). We see that under this definition the Lorentz groups consists of matrices L such that $L\overline{L} = \mathrm{Id}$. It can be shown that the identity component of this group is explicitly of the form

$$\exp\left(\begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix} \right) \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

With $A \in \mathrm{SO}(3)$, $b \in \mathbb{R}^3$. Now to find the Lie Algebra of this group, derive any curve passing through the identity at $t = 0$:

$$\frac{d}{dt}\bigg|_0 \exp\left(t \begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix} \right) \begin{pmatrix} e^{tj(\omega)} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} j(\omega) & b \\ \overline{b} & 0 \end{pmatrix} \tag{6.1}$$

So that the Lie algebra of the Lorentz group consists of matrices

$$\Lambda = \begin{pmatrix} j(\omega) & \beta \\ \overline{\beta} & 0 \end{pmatrix}$$

With ω, $\beta \in \mathbb{R}^3$. This immediately gives the Lie algebra of the Poincaré group:

$$\mathfrak{g} = \left\{ \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \mid \Lambda \in \mathfrak{so}(3,1),\ \Gamma \in \mathbb{R}^{3,1} \right\}$$

Then we immediately can identify the dual $\mathfrak{g}^*$ with pairs $x = (M, P)$, along with the dual pairing

$$\langle x, Z \rangle = \frac{1}{2}\mathrm{Tr}(\overline{M}\Lambda) + \overline{P}\Lambda$$

Computing the Coadjoint action. Note that G of course acts by matrix multiplication on $\mathfrak{g}^*$. Then, we begin by computing

(6.2)
$$\begin{aligned}\left\langle \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ P \end{pmatrix}, \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} M \\ P \end{pmatrix}, \begin{pmatrix} \overline{L} & -\overline{L}C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} M \\ P \end{pmatrix}, \begin{pmatrix} \overline{L}\Lambda L & \overline{L}(\Lambda C + \Gamma) \\ 0 & 0 \end{pmatrix} \right\rangle \\ &= \frac{1}{2}\mathrm{Tr}(M\overline{L}\Lambda L) + \overline{P}(\overline{L}(\Lambda C + \Gamma)) \\ &= \frac{1}{2}\mathrm{Tr}(M\overline{L}\Lambda L) + \overline{LP}\Lambda C + \overline{LP}\Gamma \end{aligned}$$

Now consider the term $\overline{LP}\Lambda C$. By direct computation one notes that this is precisely $\mathrm{Tr}(\Lambda C\overline{LP})$. We also see that

$$\begin{aligned} \overline{LP}\Lambda C &= \overline{\Lambda C}LP \\ &= \overline{C\Lambda}LP \qquad (6.3) \\ &= -\mathrm{Tr}(LP\overline{C}\Lambda) \end{aligned}$$

Hence, combining this with the above gives that

$$\overline{LP}\Lambda C = \frac{1}{2}\mathrm{Tr}((C\overline{LP} - LP\overline{C})\Lambda)$$

Continuing (6.2):

(6.4)
$$\begin{aligned} \frac{1}{2}\mathrm{Tr}(M\overline{L}\Lambda L) + \overline{LP}\Lambda C + \overline{LP}\Gamma &= \frac{1}{2}\mathrm{Tr}(M\overline{L}\Lambda L) + \frac{1}{2}\mathrm{Tr}((C\overline{LP} - LP\overline{C})\Lambda) + \overline{LP}\Gamma \\ &= \frac{1}{2}\mathrm{Tr}((LM\overline{L} + C\overline{LP} - LP\overline{C})\Lambda) + \overline{LP}\Gamma \end{aligned}$$

This gives our coadjoint action as

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M \\ P \end{pmatrix} = \begin{pmatrix} LM\overline{L} + C\overline{LP} - LP\overline{C} \\ LP \end{pmatrix}$$

Normal subgroup of the Poincaré group. Similar to the case for SE(3), the only normal subgroup of the Poincaré group consists of "boosts" of the form

$$\begin{pmatrix} I & C \\ 0 & 1 \end{pmatrix}$$

We see that the above computed coadjoint action is much simpler when restricted to the above normal subgroup (denoted A). We have that $g(x|_{\mathfrak{a}}) = LP$. It is clear that $\mathfrak{a}^* \cong \mathbb{R}^{3,1}$, and in order to begin classifying orbits, we will need the following

Lemma 6.1. *For* $P_i = \begin{pmatrix} p_i \\ E_i \end{pmatrix} \in \mathbb{R}^{3,1}$*, we have that* $P_1 = LP_2$ *for* $L \in SO(3,1)^o$ *if and only if* $\overline{P_1}P_1 = \overline{P}_2P_2$ *and* $E_1E_2 > 0$ *when* $\overline{P}_iP_i \geqslant 0$*.*

Proof. " $\implies$ " : Assume first that $P_1 = LP_2$. By definition of the Lorentz group, we have that $\overline{P}_1P_1 = \overline{LP_2}LP_2 = \overline{P_2}P_2$.

Assume now that $\overline{P_2}P_2 > 0$, so that without loss of generality we can assume $\overline{P_2}P_2 = 1$. If $E_2 > 0$, then $E_1 > (1 + ||p_1||^2)^{1/2} > 0$ or $E_1 < -(1 + ||p_1||^2)^{1/2} < 0$. We want to argue that the latter case is impossible. However, using connectedness, there is a path $P(t)$, $t \in [0,1]$ connecting P_1 and P_2, which projects onto a path between E_1 and E_2. Everywhere along this path, we see that $\overline{P(t)}P(t) > 0$. If $E_1 < 0$ at any $P(t_0)$, then we would see that $\overline{P(t_0)}P(t_0) < 0$, which is a contradiction. Hence, $E_1 > 0$ as well. The case for $E_2 < 0$ is nearly identical. When $\overline{P_2}P_2 = 0$ and $E_2 > 0$, we again see that $E_1 = -||p_1||$ or $E_1 = ||p_1||$. Applying an argument similar to the above,

Now we can prove the converse. We will have multiple cases to consider. Case 1: Suppose first that $\overline{P_2}P_2 = 1 > 0$ and $E_2 > 0$. We can complete this to a basis $\{P_2,\ U,\ V,\ W\}$ with $\overline{U}U = \overline{V}V = \overline{W}W = -1$ and $B = (U\ V\ W\ P_2)$ such that $\overline{B}B = I$. Then

$$B \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_2$$

An identical argument shows that there exists another matrix B' with

$$B' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_1$$

So that $P_1 = B'\overline{B}L_2$. Similarly, when $\overline{P_2}P_2 = 1 > 0$ and $E_2 < 0$, we can instead connect P_1 and P_2 to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ to conclude that $P_1 = LP_2$ for some $L \in \mathrm{SO}(3,1)^o$.

Case 2: Assume $\overline{P_2}P_2 = -1 < 0$. Again, complete to a basis $\{P_2,\ U,\ V,\ W\}$ with $\overline{U}U = \overline{V}V = \overline{W}W = -1$ and $B = (U\ V\ P_2\ W)$ such that $\overline{B}B = I$. Then

$$B\begin{pmatrix} e_3 \\ 0 \end{pmatrix} = P_2$$

And, finding B' for P_1 with

$$B'\begin{pmatrix} e_3 \\ 0 \end{pmatrix} = P_1$$

We see that $L_1 = B'\overline{B}P_2$.

Case 3: $\overline{P_2}P_2 = 0$ and $E_2 > 0$. We can find $A \in \mathrm{SO}(3)$ such that $E_2 A e_3 = p_2$, so that

$$E_2\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_3 \\ 1 \end{pmatrix} = P_2$$

And by identical reasoning as in the previous cases, we see that $P_1 = LP_2$ for some $L \in \mathrm{SO}(3,1)^o$. When $\overline{P_2}P_2 = 0$ and $E_2 < 0$, we apply the above argument for the vector $\begin{pmatrix} e_3 \\ -1 \end{pmatrix}$ instead. Finally, if $\overline{P_2}P_2 = 0$ and $E_2 = 0$, we see that $P_2 = 0$ identically, and hence so does P_1. $\square$

The proof of the converse in the above now allows us to begin our classification of orbits.

Corollary 6.2 (of proof)**.** *There is a cross section $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$ for the coadjoint action of G on $\mathfrak{a}^*$, where*

$$S_1 = \left\{ m\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid m > 0 \right\} \quad \textit{(Timelike)}$$

$$S_2 = \left\{ m\begin{pmatrix} 0 \\ -1 \end{pmatrix} \mid m > 0 \right\}$$

$$S_3 = \left\{ n \begin{pmatrix} e_3 \\ 0 \end{pmatrix} \mid n > 0 \right\} \quad \textit{(Spacelike)}$$

$$S_4 = \begin{pmatrix} e_3 \\ 1 \end{pmatrix} \quad \textit{(Lightlike)}$$

$$S_5 = \begin{pmatrix} e_3 \\ -1 \end{pmatrix}$$

$$S_6 = \{0\}$$

7. Computing Stabilizers for each Cross Section

We begin by finding the most general element of $\mathrm{SO}(3,1)^o$ in a more explicit form:

Proposition 7.1. *Every element of the Lorentz group is of the form*

$$L = \begin{pmatrix} A - u\overline{u}A + \cosh(b)u\overline{u}A & \sinh(b)u \\ \sinh(b)\overline{u} & \cosh(b) \end{pmatrix}$$

Where $A \in SO(3)$, $u \in \mathbb{R}^3$ is a unit vector, and $b \geqslant 0$.

Proof. By the opening dicussion on the Poincaré group, we already have that

$$L = \exp\left(\begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix} \right) \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

It remains to compute the exponential term. Set $||c|| := b$, so that $c = bu$ for from unit vector u. Then,

$$\begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}^3 = b^2 \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}$$

And similarly,

$$\begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}^4 = b^2 \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}^2$$

Now, setting $B := \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix}$, we have:

(7.1)

$$\begin{aligned}
\exp(B) &= \sum_{n=0}^{\infty} \frac{B^n}{n!} \\
&= I + \sum_{n=0}^{\infty} \frac{B^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{B^{2n}}{(2n)!} \\
&= I + \sum_{n=0}^{\infty} \frac{b^{2n}}{(2n+1)!} B + \sum_{n=1}^{\infty} \frac{b^{2n-2}}{(2n)!} B^2 \\
&= I + \frac{\sinh(b)}{b} B + \frac{\cosh(b)-1}{b^2} B^2 \\
&= \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \sinh(b)u \\ \sinh(b)\overline{u} & 0 \end{pmatrix} + \begin{pmatrix} \cosh(b)u\overline{u} - u\overline{u} & 0 \\ 0 & \cosh(b)-1 \end{pmatrix} \\
&= \begin{pmatrix} (\mathbf{1} - u\overline{u})\cosh(b)u\overline{u} & \sinh(b)u \\ \sinh(b)\overline{u} & \cosh(b) \end{pmatrix}
\end{aligned}$$

Now, take the product:

$$\begin{pmatrix} (\mathbf{1} - u\overline{u})\cosh(b)u\overline{u} & \sinh(b)u \\ \sinh(b)\overline{u} & \cosh(b) \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A - u\overline{u}A + \cosh(b)u\overline{u}A & \sinh(b)u \\ \sinh(b)\overline{u} & \cosh(b) \end{pmatrix}$$

□

The above explicit form will allow us to compute the stabilizers for each part of the cross section much more easily, and the next definition and proposition will simplify our work for the induction step.

Definition 7.2. Let $X_i = \begin{pmatrix} r_i \\ t_i \end{pmatrix}$, $i = 1, 2$ in Minkowski space. Define

$$j(X_1)(X_2) := \begin{pmatrix} j(r_2t_1 - r_1t_2) & r_1 \times r_2 \\ \overline{r_1 \times r_2} & 0 \end{pmatrix}$$

Proposition 7.3. *In the above definition's notation,*

$$X_1\overline{X_2} - X_2\overline{X_1} = \begin{pmatrix} j(r_1 \times r_2) & r_1t_2 - r_2t_1 \\ \overline{r_1t_2 - r_2t_1} & 0 \end{pmatrix}$$

And for any $L \in SO(3,1)^o$,

$$Lj(X_1)(X_2)\overline{L} = j(LX_1)(LX_2)$$

Stabilizer for S_1 and S_2. Recalling our definitions for S_1 and S_2, note that stabilizers of their elements will clearly coincide, so it suffices to find just the stabilizer for elements of S_1. We have:

$$\begin{pmatrix} A - u\overline{u}A + \cosh(b)u\overline{u}A & \sinh(b)u \\ \sinh(b)\overline{u} & \cosh(b) \end{pmatrix} \begin{pmatrix} 0 \\ m \end{pmatrix} = \begin{pmatrix} m\sinh(b)u \\ m\cosh(b) \end{pmatrix}$$

From which we immediately deduce that $b = 0$, and hence the stabilizer consists of matrices

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \quad A \in \mathrm{SO}(3)$$

So that the above is clearly isomorphic to SO(3).

Stabilizer for S_3. For the stabilizers of the spacelike and lightlike particles, we will need more sophisticated methods. First note that in the notation of 3.7, $G/H \cong G(p)$, $p \in S_3$. We can already see that this is a connected hyperboloid (being the level set $\overline{P}P = k < 0$, which is homeomorphic to $\mathbb{R}^3$. This is simply connected, which is to say its fundamental group is trivial. We have the short exact sequence (in multiplicative notation)

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

Which gives rise to the long exact homotopy sequence

$$\cdots \longrightarrow \pi_1(G/H) \longrightarrow \pi_0(H) \longrightarrow \pi_0(G) \longrightarrow \pi_0(G/H) \longrightarrow 1$$

Since G is defined to be the identity component, $\pi_0(G)$ is trivial as well. Exactness yields that $\pi_0(H) = 1$ (since simple connectedeness gives $\pi_1(G/H) = 1$). This means that we can find the stabilizer of p in $\mathfrak{h}$ and then recover all of H via exponentiation, which is a considerably easier task.

Proposition 7.4. *Given $p \in S_3$, the stabilizer H consists of matrices of the form*

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\cosh(|b|)u_1^2 - u_1^2 & (\cosh(|b|)-1)u_1u_2 & 0 & \sinh(|b|)u_1 \\ (\cosh(|b|)-1)u_1u_2 & 1+\cosh(|b|)u_2^2 - u_2^2 & 0 & \sinh(|b|)u_2 \\ 0 & 0 & 1 & 0 \\ \sinh(|b|)u_1 & \sinh(|b|)u_2 & 0 & \cosh(|b|) \end{pmatrix}$$

where $u = (u_1, u_2)$ is a unit vector and $|b|$ is such that $b = |b|u$.

Proof. Working in the Lie algebra, we only need to find the set of all Λ such that $\Lambda(p) = 0$. Explicitly,

$$\begin{pmatrix} j(\ell) & b \\ \overline{b} & 0 \end{pmatrix} \begin{pmatrix} e_3 \\ 0 \end{pmatrix} = 0$$

From which we immediately find that $\ell \times e_3 = 0$, and $\langle b, e_3 \rangle = 0$. Hence, this consists of all matrices of the form

$$\begin{pmatrix} j(\alpha e_3) & b \\ \overline{b} & 0 \end{pmatrix}$$

with $\alpha \in \mathbb{R}$ and $\langle b, e_3 \rangle = 0$. Now we want to compute the matrix exponential of the above. Letting b denote just the 2-dimensional vector (b_1, b_2), we see

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} b\overline{b} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |b|^2 \end{pmatrix}$$

And similarly

$$\begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}^3 = |b|^2 \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}$$

Allowing easy computation of our exponential as

$$\begin{pmatrix} I - u\overline{u} + \cosh(|b|)u\overline{u} & 0 & \sinh(|b|)u \\ 0 & 1 & 0 \\ \sinh(|b|)\overline{u} & 0 & \cosh(|b|) \end{pmatrix}$$

Exponentiation of the matrix

$$\begin{pmatrix} j(\alpha e_3) & 0 \\ 0 & 0 \end{pmatrix}$$

is standard. Taking the product of these yields the desired stabilizer H. $\square$

Corollary 7.5. *$H \cong SO(2,1)$ (and hence, by the $2-1$ covering $SL(2,\mathbb{R}) \to SO(2,1)$, $H(x|_{\mathfrak{h}}$ can be considered an orbit of $SL(2,\mathbb{R})$).*

Stabilizers for S_4 and S_5. We now consider the case of the lightlike particles. By identical reasoning as in the previous case (via the long exact homotopy sequence), we can deduce that H consists of only one component by noting that $G/H \cong \mathbb{R}\backslash\{0\}$ (the top/bottom half of a cone). Hence we can employ the same strategy of finding the Lie algebra of our stabilizer and then exponentiating.

Proposition 7.6. *Given $p \in S_4 \cup S_5$, the stabilizer H consists of matrices of the form*

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -b_1 & b_1 \\ 0 & 1 & -b_2 & b_2 \\ b_1 & b_2 & 1-|b|^2/2 & |b|^2/2 \\ b_1 & b_2 & -|b|^2/2 & 1+|b|^2/2 \end{pmatrix}$$

with $b = (b_1, b_2) \in \mathbb{R}^2$.

Proof. We only need to consider the case for $p \in S_4$. Similar to the spacelike case, we compute

$$\begin{pmatrix} j(\ell) & b \\ \bar{b} & 0 \end{pmatrix} \begin{pmatrix} e_3 \\ 0 \end{pmatrix} = 0$$

To find that $\langle b, e_3 \rangle = 0$, and $e_3 \times \ell = b$, that is, $\ell = (\ell_1, -b_1, b_2)$ when $b = (b_1, b_2, 0)$, so that our Λ element is of the form

$$\begin{pmatrix} 0 & -\ell_1 & -b_1 & b_1 \\ \ell_1 & 0 & -b_2 & b_2 \\ b_1 & b_2 & 0 & 0 \\ b_1 & b_2 & 0 & 0 \end{pmatrix}$$

Now we again view b as a vector in 2 dimensions and consider the block matrix

$$\begin{pmatrix} 0 & -b & b \\ \overline{b} & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}$$

We see

$$\begin{pmatrix} 0 & -b & b \\ \overline{b} & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -|b|^2 & |b|^2 \\ 0 & -|b|^2 & |b|^2 \end{pmatrix}$$

And

$$\begin{pmatrix} 0 & -b & b \\ \overline{b} & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}^3 = 0$$

Hence we only need compute 3 terms of the exponential series, yielding

$$\begin{pmatrix} 1 & 0 & -b_1 & b_1 \\ 0 & 1 & -b_2 & b_2 \\ b_1 & b_2 & 1 - |b|^2/2 & |b|^2/2 \\ b_1 & b_2 & -|b|^2/2 & 1 + |b|^2/2 \end{pmatrix}$$

And the statement of the proposition follows immediately. □

Corollary 7.7. *$H(x|_{\mathfrak{h}})$ is an orbit of $E(2)$, so that we have that $H(x|_{\mathfrak{h}}) \cong TS^1$.*

Stabilizer of S_6.

Proposition 7.8. *For $0 \in S_6$, $H = SO(3,1)^o$*

8. Computing $\mathrm{Ind}_{\mathrm{SO}(3)}^{\mathrm{SO}(3,1)^o \times \mathbb{R}^{3,1}} S^2$

We first set $N = T^*G \times S^2 \cong G \times \mathfrak{g}^* \times S^2$. We will not worry about computing the 2-form here. Note that S^2 has an H-space structure by the beginning examples for the coadjoint orbit of SO(3), with moment map merely being the inclusion. Using the notation for the definition of Symplectic induction, we see:

$$\psi^{-1}(0) = \left\{ \left(\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j(su) & g \\ \overline{g} & 0 \end{pmatrix}, m \begin{pmatrix} 0 \\ 1 \end{pmatrix}, su \right) \mid \quad \begin{matrix} u \in S^2, \ L \in \mathrm{SO}(3,1) \\ C \in \mathbb{R}^{3,1}, \ g \in \mathbb{R}^3 \end{matrix} \right\}$$

Define $I := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and consider the second element of the above 4-tuples. We can decompose it as

$$\begin{pmatrix} j(su) & g \\ \overline{g} & 0 \end{pmatrix} = s \begin{pmatrix} j(u) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & g \\ \overline{g} & 0 \end{pmatrix}$$

Now associate to u the 4-vector $U := \begin{pmatrix} u \\ 0 \end{pmatrix}$ and g the 4-vector $G := \begin{pmatrix} g/m \\ 0 \end{pmatrix}$. Observe that $\overline{I}I = 1$, $\overline{I}U = 0$, and $\overline{U}U = -1$, and

$$\begin{pmatrix} j(su) & g \\ \overline{g} & 0 \end{pmatrix} = sj(I)(U) + m(G\overline{I} - I\overline{G})$$

Then, letting ℓ denote an element of the Poincaré group, every tuple of $\psi^{-1}(0)$ is of the form $(\ell, sj(I)(U) + m(G\overline{I} - I\overline{G}), mI, su)$. Letting $h \in \mathrm{SO}(3)$, h will act on the above tuples via

$$h(\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mhI, su) = (\ell\overline{h}, sj(I)(hU)+m(hG\overline{I}-I\overline{hG}), mI, shu)$$

Also, when $\ell = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$ is an element of the Lorentz group, we have moment map $\phi : \psi^{-1}(0) \to \mathfrak{g}^*$ with

$$\phi\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big) = (sj(LI)(LU)+m\big((LG+C)\overline{LI}-LI\overline{(LG+C)}, mLI)$$

This allows us the following

Theorem 8.1. *Let V denote the space of triples*

$$\begin{pmatrix} X \\ T \\ J \end{pmatrix}$$

with T and J 4-vectors such that $\overline{T}T = 1$, $\overline{T}J = 0$, $\overline{J}J = -1$, and X a point in Minkowski space. Then there is a G-equivariant surjection

$\Phi : \psi^{-1}(0) \to V$ *such that the following diagram commutes:*

$$\begin{tikzcd} \psi^{-1}(0) \arrow[r] \arrow[d, "\Phi"] & N \arrow[d, "\phi"] \\ V \arrow[r, "\mu"] & \mathfrak{g}^* \end{tikzcd}$$

where

$$\mu\left(\begin{pmatrix} X \\ T \\ J \end{pmatrix}\right) = \big(sj(T)(J) + m(X\overline{T} - T\overline{X}), mT\big)$$

In particular,

$$\Phi(\big((\ell, sj(I)(U) + m(G\overline{I} - I\overline{G}), mI, su)\big) = \begin{pmatrix} LG + C \\ LI \\ LU \end{pmatrix}$$

The proof of the above will be done in several steps. We must first define how the Poincaré group will act on V. This definition is very natural, however, as we merely have

$$\begin{pmatrix} L' & C' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X \\ T \\ J \end{pmatrix} = \begin{pmatrix} L'X + C \\ L'T \\ L'J \end{pmatrix}$$

The advantage to the above characterization is that we can instead consider $\mathrm{Ind}_{\mathrm{SO}(3)}^{\mathrm{SO}(3,1)} S^2$ as the quotient of V by a suitable equivalence, which will be amde explicit later.

Proof. **Surjectivity of Φ:** To show surjectivity of Φ, let $\begin{pmatrix} X \\ T \\ J \end{pmatrix} \in V$. Since $\overline{T}T = 1$, we can find $L \in \mathrm{SO}(3,1)$ such that $LI = T$. Now, find $C \in \mathbb{R}^{3,1}$ such that $\overline{L}X - \overline{L}C = \begin{pmatrix} b \\ 0 \end{pmatrix} := B$ for $b \in \mathbb{R}^3$. By definition of the Lorentz group, L must preserve the Minkowski metric. We already know $\overline{T}J = 0$, and since $LI = T$, we deduce that $\overline{L}J = \begin{pmatrix} u \\ 0 \end{pmatrix} := U$ for $u \in \mathbb{R}^3$, $|u| = 1$. Hence,

$$\Phi\left(\left(\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}, sj(I)(U) + m\big(B\overline{I} - I\overline{B}\big), mI, su\right)\right) = \begin{pmatrix} X \\ T \\ J \end{pmatrix}$$

Showing surjectivity.

Commutativity of the Diagram: Let us factor through the top right corner. Then ϕ is merely the action map on $\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big)$. We compute:

$$\begin{aligned}
&\phi\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big)\\
&=\ell\left(\begin{pmatrix} sj(I)(U)+m(G\overline{I}-I\overline{G}) \\ mI \end{pmatrix}\right)\\
&=\begin{pmatrix} sLj(I)(U)\overline{L}+m(LG\overline{LI}-LI\overline{LG})+m(C\overline{LI}-LI\overline{C} \\ mLI \end{pmatrix}\\
&=\begin{pmatrix} sj(LI)(LU)+m((LG+C)\overline{LI}-LI\overline{(LG+C)}) \\ mLI \end{pmatrix}
\end{aligned} \tag{8.1}$$

Likewise, we can compute travelling along the bottom left corner as

$$\begin{aligned}
&\mu\circ\Phi\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big)\\
&=\mu\left(\begin{pmatrix} LG+C \\ LI \\ LU \end{pmatrix}\right)\\
&=\begin{pmatrix} sj(LI)(LU)+m((LG+C)\overline{LI}-LI\overline{(LG+C)}) \\ mLI \end{pmatrix}
\end{aligned} \tag{8.2}$$

So the diagram does commute.

***G*-equivariance of all maps involved:** Equivariance over the top right corner is obvious by construction. Along the bottom corner, let ℓ' be an element of our Poincaré group. Then,

$$\begin{aligned}
&\Phi(\ell'\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big))\\
&=\begin{pmatrix} L'LG+L'C+C' \\ L'LI \\ L'LU \end{pmatrix}\\
&=\begin{pmatrix} L' & C' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} LG+C \\ LI \\ LU \end{pmatrix}\\
&=\ell'\Phi\big((\ell, sj(I)(U)+m(G\overline{I}-I\overline{G}), mI, su)\big)
\end{aligned} \tag{8.3}$$

Equivariance of μ is similarly trivial.

$\square$

Define U to be the quotient of V by the relation

$$\begin{pmatrix} X \\ T \\ J \end{pmatrix} = \begin{pmatrix} X' \\ T' \\ J' \end{pmatrix} \iff \begin{matrix} X - X' = \lambda T \\ T = T',\ J = J' \end{matrix}$$

We can formally denote such elements as $\begin{pmatrix} X + \mathbb{R}T \\ T \\ J \end{pmatrix}$.

Theorem 8.2. *$Ind_{SO(3)}^{SO(3,1)^o \times \mathbb{R}^{3,1}} S^2$ has the structure of a Hamiltonian G-space via the identification $\Phi : \psi^{-1}(0) \to V \to U \cong Ind_{SO(3)}^{SO(3,1)^o \times \mathbb{R}^{3,1}} S^2$. Given $x := \begin{pmatrix} X + \mathbb{R}T \\ T \\ J \end{pmatrix} \in U$, set $\Omega := j(T)(J)$. Then we have a symplectic 2-form*

$$\sigma(\delta x, \delta' x) = -s\, Tr(\delta\Omega \cdot \Omega \cdot \delta'\Omega) + m(\overline{\delta X}\delta' T - \overline{\delta' X}\delta T)$$

Where $G = SO(3,1)^o \times \mathbb{R}^{3,1}$ acts on U via

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X + \mathbb{R}T \\ T \\ J \end{pmatrix} = \begin{pmatrix} LX + C + \mathbb{R}LT \\ LT \\ LJ \end{pmatrix}$$

Proof. The proof of this follows mostly from the work of Souriau and 3.7. $\square$

Setting $s = 0$ in the above, we are inducing a manifold from the origin. This physically represents a particle of mass m with 0 spin. Reasoning identical to the above leads to the simpler structure of the following:

Corollary 8.3. *Let V denote the space of points of the form $\begin{pmatrix} X + \mathbb{R}T \\ T \end{pmatrix}$ with X a point of Minkowski space and I a 4-vector with $\overline{I}I = 1$. Then,*

$V \cong Ind_{SO(3)}^{SO(3,1)^o \times \mathbb{R}^{3,1}}\{0\}$, where $\psi^{-1}(0)$ consists of all tuples $(\ell, m(G\overline{I} - I\overline{G}), mI)$ with

$$\Phi\big((\ell, m(G\overline{I} - I\overline{G}), mI)\big) = \begin{pmatrix} LG + C + \mathbb{R}LI \\ LI \end{pmatrix}$$

(same definitions as above).

We have symplectic 2-form

$$\sigma(\delta x, \delta' x) = m(\overline{\delta X}\delta' T - \overline{\delta' X}\delta T)$$

and G-action

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X + \mathbb{R}T \\ T \end{pmatrix} = \begin{pmatrix} LX + C + \mathbb{R}LT \\ LT \end{pmatrix}$$

9. Computing $\mathrm{Ind}_{\mathrm{E}(2)}^{\mathrm{SO}(3,1)\times\mathbb{R}^{3,1}}\{se_3\}$

We now proceed to discuss the case of lightlike particles and map them onto a framework similar to that given above. We begin by computing $\psi^{-1}(0)$ for se_3 for $s \in \mathbb{R}$ (that is, we are considering the degenerate cylinder along the z-axis). It is straightforward to find

$$\psi^{-1}(0) = \left\{ \left(\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} j(se_3) & b \\ \overline{b} & 0 \end{pmatrix}, \begin{pmatrix} e_3 \\ 1 \end{pmatrix}, se_3 \right) \mid \begin{array}{c} s \in \mathbb{R},\ L \in \mathrm{SO}(3,1) \\ C \in \mathbb{R}^{3,1},\ b \in \mathbb{R}^3,\ \langle b, e_3 \rangle = 0 \end{array} \right\}$$

Now, redefine s to be its absolute value and take the sign of s to be denoted by χ (the *helicity*). Denote by E the matrix $\begin{pmatrix} j(e_3) & 0 \\ 0 & 0 \end{pmatrix}$. Recall that $b \in \mathbb{R}^3$ is of the form $(b_1, b_2, 0)$. Associate to b the 4-vector $\begin{pmatrix} b \\ 0 \end{pmatrix}$. Define $I := \begin{pmatrix} e_3 \\ 1 \end{pmatrix}$. One easily verifies

$$\begin{pmatrix} 0 & b \\ \overline{b} & 0 \end{pmatrix} = B\overline{I} - I\overline{B}$$

So we have a decomposition

$$\begin{pmatrix} j(se_3) & b \\ \overline{b} & 0 \end{pmatrix} = s\chi E + B\overline{I} - I\overline{B}$$

Leading to the following:

Theorem 9.1. *Let V denote the space of triples*

$$\begin{pmatrix} X \\ T \\ \Omega \end{pmatrix}$$

with T a 4-vector such that $\overline{T}T = 0$, $\Omega I = 0$ and $\frac{1}{2}Tr(\overline{\Omega}\Omega) = 1$ and X a point in Minkowski space. Then there is a G-equivariant surjection $\Phi : \psi^{-1}(0) \to V$ such that the following diagram commutes:

$$\begin{tikzcd} \psi^{-1}(0) \arrow[r] \arrow[d, "\Phi"] & N \arrow[d, "\phi"] \\ V \arrow[r, "\mu"] & \mathfrak{g}^* \end{tikzcd}$$

where

$$\mu\left(\begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} \right) = \big(s\chi\Omega + \eta(X\overline{T} - T\overline{X}), \eta T\big)$$

(η denotes the sign of the energy). In particular,

$$\Phi(\big((\ell, s\chi E + (B\overline{I} - I\overline{B}), I, se_3)\big) = \begin{pmatrix} LB + C \\ LI \\ LE\overline{L} \end{pmatrix}$$

Before the proof, note that we have the action

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} = \begin{pmatrix} LX + C \\ LT \\ L\Omega\overline{L} \end{pmatrix}$$

of the Poincaré group on V.

Proof. **Surjectivity of Φ:** Let $\begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} \in V$. We can find $L \in \mathrm{SO}(3,1)$ such that $LI = T$. Since L preserves our Minkowski metric, we deduce that if $J := \frac{1}{2}\begin{pmatrix} e_3 \\ -1 \end{pmatrix}$, then $\Omega = j(LI)(LJ) = Lj(I)(J)\overline{L}$. By direct computation, one easily verifies that $j(I)(J) = E$, so $\Omega = LE\overline{L}$.

Finally, we can obviously choose C such that $\overline{L}X - C = \begin{pmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{pmatrix}$ for some real numbers $b_1,\ b_2$. Hence, defining this to be B, we get surjectivity of Φ.

The proof of commutativity and equivariance follows almost identically to the timelike case.

□

We now define a quotient U of V by the following equivalence:

$$\begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} = \begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} \iff \begin{matrix} \exists Z \text{ s.t } \overline{I}Z = 0 \\ X' = X + \eta\chi s Z \\ I' = I \\ \Omega' = \Omega + Z\overline{I} - I\overline{Z} \end{matrix}$$

Then, we have by Souriau and the above:

Theorem 9.2. *$Ind_{E(2)}^{SO(3,1)^o \times \mathbb{R}^{3,1}}\{se_3\}$ has the structure of a Hamiltonian G-space via the identification $\Phi : \psi^{-1}(0) \to V \to U \cong Ind_{E(2)}^{SO(3,1)^o \times \mathbb{R}^{3,1}}\{se_3\}$. Given $x := \begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} \in U$, we have a symplectic 2-form*

$$\sigma(\delta x, \delta' x) = -\chi s\, Tr(\delta\Omega \cdot \Omega \cdot \delta'\Omega) + \eta(\overline{\delta X}\delta' T - \overline{\delta' X}\delta T)$$

Where $G = SO(3,1)^o \times \mathbb{R}^{3,1}$ acts on U via

$$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ T \\ \Omega \end{pmatrix} = \begin{pmatrix} LX + C \\ LT \\ L\Omega\overline{L} \end{pmatrix}$$

Similar to the case for a particle with mass, we can set $s = 0$ and induce from the origin. This corresponds to a massless particle with 0 spin, and the Ω term drops out of the computations.

We can let V be the space with tuples $\begin{pmatrix} X \\ T \end{pmatrix}$ with $\overline{T}T = 0$. Quotient by the equivalence (we will define this quotient to be U)

$$\begin{pmatrix} X \\ T \end{pmatrix} = \begin{pmatrix} X' \\ T' \end{pmatrix} \iff \begin{matrix} \exists Z \text{ s.t } \overline{Z}T = 0 \\ X' = X + \lambda Z, \ \lambda \in \mathbb{R} \end{matrix}$$

It turns out that this is $\mathrm{Ind}_{\mathrm{E}(2)}^{\mathrm{SO}(3,1)^o \times \mathbb{R}^{3,1}}\{0\}$. $\psi^{-1}(0)$ is already of the form $(\ell, B\overline{I} - I\overline{B}, E)$, with map $\Phi : \psi^{-1}(0) \to V$ defined as expected:

$$\Phi(\ell, B\overline{I} - I\overline{B}, I) = \begin{pmatrix} LB + C \\ LI \end{pmatrix}$$